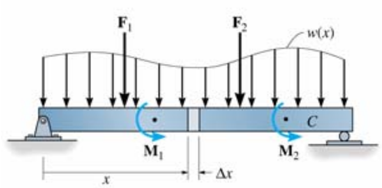
Introduction to Mechanics of Materials

Lecture 16, October 10, 2003

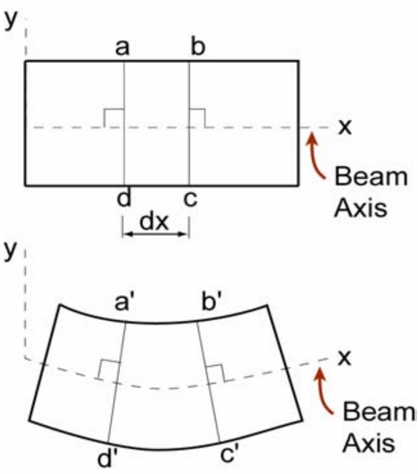
Bending

Flexure Formula, Stress Concentrations

Geometry of Beam Deflection



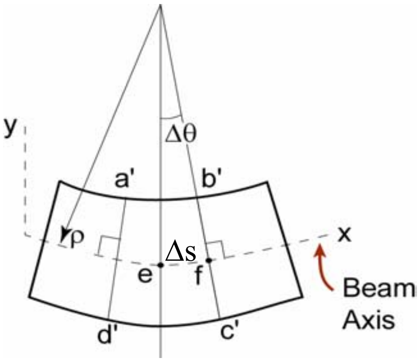
Planes *ad* and *bc* ⊥ to beam axis



Planes *a'd'* and *b'c'* ⊥ to beam axis

Plane sections through a beam taken normal to its axis, *remain plane* after the beam is subjected to bending.

Geometry of Beam Deflection: Rectangular cross section, elastic deformation, PURE bending (no transverse shear)



Deforms to the shape of an arc

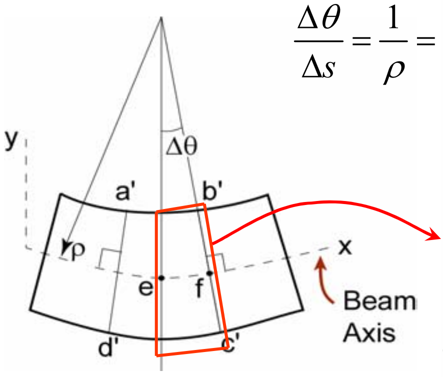
$$ef = \Delta s = \rho\Delta\theta$$

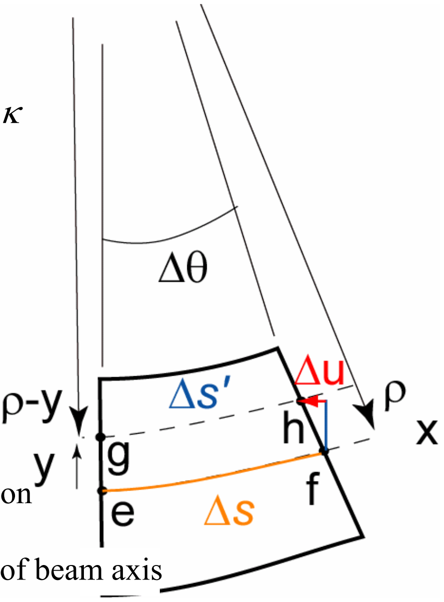
$$\frac{\Delta\theta}{\Delta s} = \frac{1}{\rho} = \kappa$$

κ = reciprocal of ρ
= reciprocal of curvature
Curvature is constant for pure bending

Geometry of Beam Deflection: Rectangular cross section, elastic deformation, PURE b

$$\frac{\Delta\theta}{\Delta s} = \frac{1}{\rho} = \kappa$$

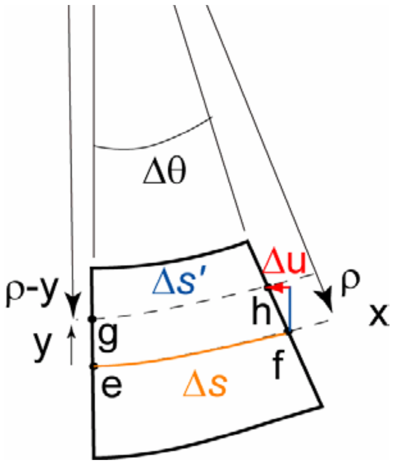




Let $u = u(x,y)$, be stretch in x-direction
(Positive in x-direction)
$u(x,0) = 0$ requirement for location of beam axis

Geometry of Beam Deflection: Rectangular cross section, elastic deformation, PURE bending

Let $u = u(x,y)$, be stretch in x-direction (Positive in x-direction)



$$\frac{\Delta\theta}{\Delta s} = \frac{1}{\rho} = \kappa$$

$$\Delta s' + (-\Delta u) = \Delta s$$

Since Δu pos. to the right

$$\Delta u = \Delta s' - \Delta s$$

$$\Delta u = (\rho - y)\Delta\theta - \rho\Delta\theta$$

$$\Delta u = -y\Delta\theta$$

$$\frac{\Delta u}{\Delta s} = -y\frac{\Delta\theta}{\Delta s} = -y\kappa$$

$$\frac{\Delta u}{\Delta s} = -y\kappa \quad \Rightarrow \quad \boxed{\frac{du}{dx} = -y\kappa}$$

$$\Delta\theta \to 0 \quad \begin{cases} \Delta u \to du \\ \Delta s \to dx \end{cases}$$

The Elastic Flexure Formula

$$\frac{du}{dx} = -y\kappa$$

but $\frac{du}{dx} = \varepsilon_x$ normal strain

Basic Kinematic Hypothesis for the Flexure Theory

$$\boxed{\varepsilon_x = -y\kappa}$$

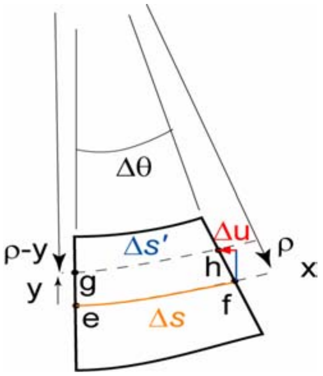
Hooke's Law $\sigma_x = E\varepsilon_x$

$$\Rightarrow$$

$$\boxed{\sigma_x = -Ey\kappa}$$

constant

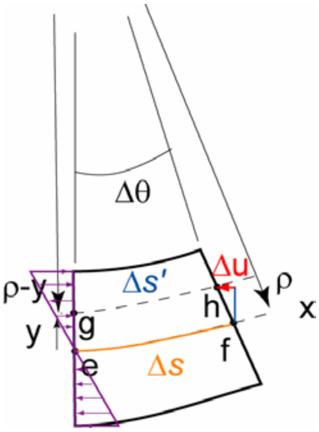
The Elastic Flexure Formula, #1



Flexure Formula

$$\sigma_x = -Ey\kappa$$

Normal, longitudinal stresses



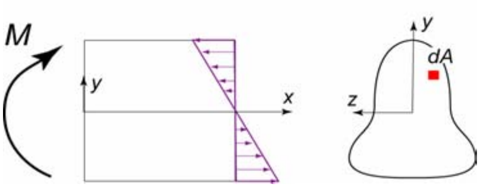
$y < 0 \quad \sigma_x > 0$ Tensile

$y = 0 \quad \sigma_x = 0$ No stress or strain along x axis

$y > 0 \quad \sigma_x < 0$ Compressive

Force Equilibrium of a Cross Section

$$\sigma_x = -Ey\kappa$$



$$\Sigma F_x = 0 \Rightarrow \int_A \sigma_x dA = 0$$

constant

$$\int_A -E\kappa y dA = 0$$

$$-E\kappa \int_A y dA = 0 \quad \boxed{\int_A y dA} = 0$$

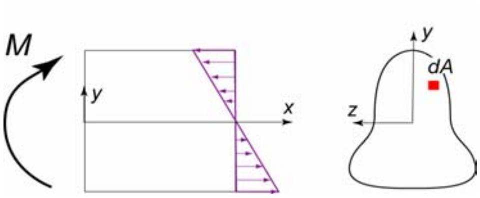
By definition: $\int_A y dA = \overline{y}A$ where $\overline{y}$ = location of centroid

in this case $\int_A y dA = 0$

thus, beam axis (x-axis) must coincide with the centroid

Bending Moment Equilibrium

$$\boxed{\sigma_x = -Ey\kappa}$$



$$\Sigma M_z = 0 \Rightarrow$$

$$-M + \int_A |\sigma_x| y dA = 0$$

$$M = \int_A Ey\kappa y dA$$

$$M = E\kappa \boxed{\int_A y^2 dA}$$

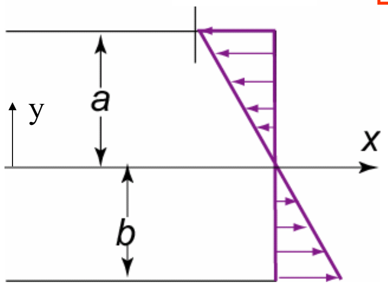
Area Moment of Inertia $I_z = \int_A y^2 dA$

$M = E\kappa I_z$ or $\boxed{\kappa = \frac{M}{EI_z}}$ Basic relationship for curvature

Flexure Formula $\boxed{\sigma_x = -\frac{My}{I_z}}$

Stress Distribution

$$\boxed{\sigma_x = -\frac{M_z y}{I_z}}$$



$y < 0$ $\sigma_x > 0$ Tensile

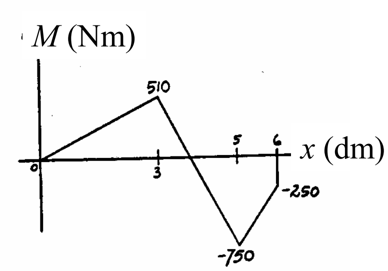
$y = 0$ $\sigma_x = 0$ No stress along x axis

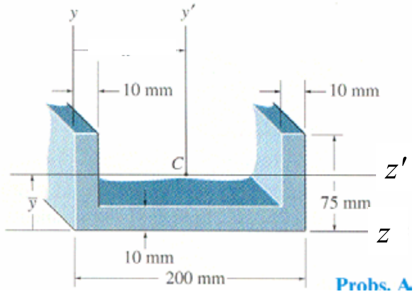
$y > 0$ $\sigma_x < 0$ Compressive

Let max(a,b) = c $\boxed{\sigma_{\max} = \left|\frac{Mc}{I_z}\right|}$

Example

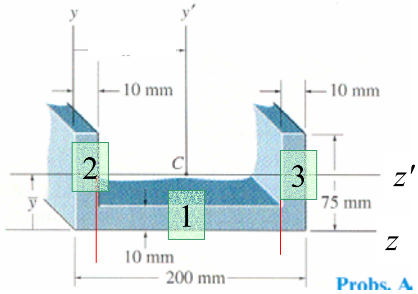
- Given:
  - Bending Moment Diagram
  - Cross Section
- Determine:
  - Max Bending Stresses over cross section





Solution

1. Determine I_z



Centroid:

$$\bar{y} = \frac{\sum_{i=1..3} y_i A_i}{\sum_{i=1..3} A_i}$$

$$A = \sum_{i=1..3} A_i$$

$$A = 180(10) + 2(10)(75) = 3300 mm^2$$

$$\sum_{i=1..3} y_i A_i = 5(180)(10) + 2(75/2)(10)(75) = 65250 mm^3$$

$$\bar{y} = \frac{65250}{3300} = 19.773 mm \qquad \boxed{\bar{y} = 19.773 mm}$$

Moment of Inertia $I_z = \sum \bar{I}_i + \sum \left(A d_y^2 \right)_i$

$$I_z = \tfrac{1}{12} 180\left(10^3\right) + 180(10)(19.733 - 5)^2$$

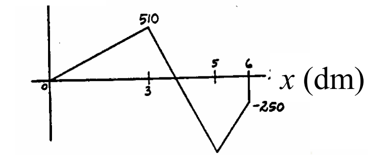
$$+2\left[\tfrac{1}{12} 10\left(75^3\right) + 10(75)(19.733 - 75/2)^2\right] \qquad \boxed{I_z = 1.58(10^6) mm^4}$$

Solution, cont

2. Determine Stresses $\sigma = -\frac{My}{I_z}$

$$\boxed{M_{\max} = 510 Nm}$$

$$\boxed{M_{\min} = -750 Nm}$$



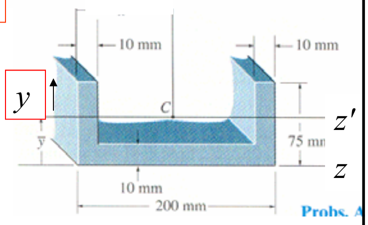
$$y_{top} = 75 - 19.773 = 55.227 mm$$

$$y_{bottom} = -19.773 mm$$

Tensile stress:

$$\sigma_{top} = -\frac{My}{I_z} = -\frac{(-750)(55.227)(10^{-3})}{1.58\left(10^{-6}\right)} = \boxed{26.21 MPa}$$

$$\sigma_{bottom} = -\frac{(510)(-19.773)(10^{-3})}{1.58\left(10^{-6}\right)} = 6.38 MPa$$



Compressive stress:

$$\sigma_{top} = -\frac{(510)(55.227)(10^{-3})}{1.58\left(10^{-6}\right)} = \boxed{-17.83 MPa}$$

$$\sigma_{bottom} = -\frac{(-750)(-19.773)(10^{-3})}{1.58\left(10^{-6}\right)} = -9.39 MPa$$

Homework Assignment due Tuesday Oct. 14

- Text: Lecture Notes and section 6-3 and 6-4
- Read *Important points*, page 289
- Read *Procedure for Analysis*, page 289
- Read Examples 6-14 to 6-17
- Do problems 6-39, 6-50, 6-61